[the simple linear regression model (SLR)]

STAT 206
10 Mar 21

DD extra OH

$$Y_i = \beta_0 + \beta_1 x_i + e_i \leftarrow \text{IID } N(0, \sigma^2)$$

$(i = 1, \ldots, n)$

8124

(1)

$\underset{\sim}{\theta} = (\beta_0, \beta_1, \sigma^2)$

$\underset{\sim}{y}_n^T = (y_1, \ldots, y_n)$

$$(Y_i \mid [SM: SLR]\, \underset{\sim}{\theta}\, x_i\, \mathcal{B}) \overset{I}{\sim} N(\underline{\beta_0 + \beta_1 x_i}, \sigma^2)$$
$$(i = 1, \ldots, n)$$

$$\underset{n \times 2}{X} = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$$

$\underset{\sim}{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$ $\underset{\sim}{e} = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$

$$\underset{n \times 1}{\underset{\sim}{y}} = \underset{n \times 2}{X} \cdot \underset{2 \times 1}{\beta} + \underset{n \times 1}{\underset{\sim}{e}}$$

p	...	ring#	ring#dummy1	~2	~3	
Y		'o'	0	1	0	1
15		'n'	1	0	0	1
8		'o'	0	1	0	1
0s		't'	0	0	1	1
		⋮				

n = 8124

$$\text{ring\#dummy1} = \begin{cases} 1 & \text{if ring\#} = \text{'n'} \\ 0 & \text{else} \end{cases}$$

$$2 \quad \begin{cases} 1 & \text{if 'o'} \\ 0 & \text{else} \end{cases}$$

$$3 \quad \begin{cases} 1 & \text{if 't'} \\ 0 & \text{else} \end{cases}$$

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_{i1} + \hat{\beta}_2 X_{i2} + \ldots + \hat{\beta}_5 X_{i5}$$

$\hat{\beta}_j$ are MLEs under model

$$Y_i = \text{———} + e_i$$

e_i: IID $N(0, \sigma^2)$

special case: only one x

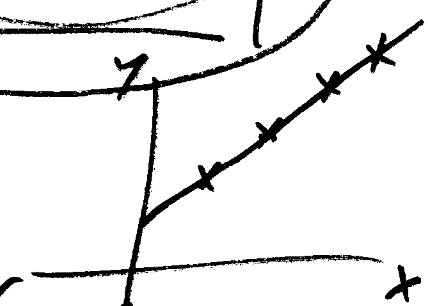
$$Y_i = \beta_0 + \beta_1 X_i + e_i$$

$$(Y_i \mid -) \overset{I}{\sim} N(\beta_0 + \beta_1 x_i, \sigma^2)$$

$$p(y \mid -) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y_i - [\beta_0 + \beta_1 x_i]}{\sigma}\right)^2\right]$$

$$\ell(\beta_0, \beta_1, \sigma \mid -) = \sigma^{-n} (2\pi)^{-\frac{n}{2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i - (\beta_0 + \beta_1 x_i)\right)^2\right]$$

$$\ell\ell(\beta_0, \beta_1, \sigma \mid -) = -n \log \sigma - \frac{n}{2} \log 2\pi \underbrace{- \frac{1}{2\sigma^2}}_{-\frac{1}{2}\sigma^{-2}} \underbrace{\sum_{i=1}^{n}\left[y_i - (\beta_0 + \beta_1 x_i)\right]^2}_{SS}$$



$\hat{\beta}_{MLE}$ minimizes this sum of squares: (ordinary) least-squares estimates (OLS)

normal equations:
$$\begin{cases} \frac{\partial}{\partial \beta_0} \ell\ell(-) = 0 \\ \frac{\partial}{\partial \beta_1} \ell\ell(-) = 0 \end{cases}$$

$$\frac{\partial}{\partial \sigma} \ell\ell(\text{—}) = -\frac{n}{\sigma} + \frac{\cancel{\sigma^{-3}}(\tilde{\varepsilon}'\tilde{\varepsilon})}{\sigma^3} = 0$$

$$0 = \frac{\tilde{\varepsilon}'\tilde{\varepsilon} - n\sigma^2}{\sigma^3}$$

$$\text{iff} \quad \sigma^2 = \hat{\sigma}^2_{MLE} = \frac{(\hat{\tilde{\varepsilon}}'\hat{\tilde{\varepsilon}})}{n}$$

1/0

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \dots + \hat{\beta}_5 x_{i5}$$

$$x_{i1} = \begin{cases} 1 & \text{if cap.shape} = \text{'c'} \\ 0 & \text{else} \end{cases}$$

what is $\hat{y}_i$ for cap.shape = 'b' (omitted group) ?

$\hat{y}_i$ for cap.shape = 'b' = $\hat{\beta}_0$ = 0.10619

$\hat{y}_i$ for cap.shape = 'b' (omitted group)

= $\bar{y}$ when cap.shape = 'b'

$\hat{y}_i$ for cap.shape = 'c'

= $\bar{y}$ when cap.shape = 'c'

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 \cdot \boxed{\begin{cases} 1 & \text{if cap.shape = 'c'} \\ 0 & \text{else} \end{cases}}$$

$$+ \hat{\beta}_2 \cdot \begin{cases} 1 & \text{if cap.shape = 'f'} \\ 0 & \text{else} \end{cases}$$

$$+ \ldots + \hat{\beta}_5 \cdot \begin{cases} 1 & \text{if cap.shape = 'x'} \\ 0 & \text{else} \end{cases}$$

$$\left(\begin{array}{c} \bar{y} \text{ when} \\ \text{cap.shape = 'c'} \end{array} \right) = \hat{\beta}_0 + \hat{\beta}_1$$

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$$\hat{\beta}_1 = \begin{pmatrix} \bar{y} \text{ when} \\ \text{cup.shape = 'c'} \end{pmatrix} - \begin{pmatrix} \bar{y} \text{ when} \\ \text{cup.shape = 'b'} \end{pmatrix}$$

$$= 1 - 0.106194$$

$$= 0.89381$$

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the $\hat{\beta}_i$ are <u>unbiased</u> for β_i

but $\hat{\sigma}^2_{MLE}$ is (badly ← when k is big biased on the low side) for σ^2

$$\hat{\sigma}^2_{MLE} = \frac{\hat{S}\hat{S}}{n} \text{ but } \hat{\sigma}^2_{unbiased} = \frac{\hat{S}\hat{S}}{n - (k+1)}$$

(k = # of xs in model)