Binomial story: (Biden vs. not Biden)

STAT 206
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$n = 1265$

$n_1 = 659$

$(N_1 \mid n\ \theta_1\ B) \sim$ Binomial(n, θ_1)

$E(N_1) = n\theta_1$

$V(N_1) = n\,\theta_1(1-\theta_1)$

$\hat{\theta}_1 = \frac{N_1}{n}$

$\hat{V}(\hat{\theta}_1) = \hat{V}\left(\frac{N_1}{n}\right)$

$= \frac{1}{n^2}\hat{V}(N_1)$

$= \frac{1}{n^2}\, n\, \hat{\theta}_1(1-\hat{\theta}_1)$

$= \frac{\hat{\theta}_1(1-\hat{\theta}_1)}{n}$

$(c \neq 0)$

$V(cX) = c^2 V(X)$

$\hat{V}(\hat{\theta}_1)$

$$\Sigma = \begin{array}{c|ccc} & \hat{\theta}_1 & \hat{\theta}_2 & \theta_3 \\ \hline \hat{\theta}_1 & \boxed{C(\hat{\theta}_1, \hat{\theta}_1)} & C(\hat{\theta}_1, \hat{\theta}_2) & C(\hat{\theta}_1, \hat{\theta}_3) \\ \hat{\theta}_2 & & C(\hat{\theta}_2, \hat{\theta}_2) & \\ \hat{\theta}_3 & & & C(\hat{\theta}_3, \hat{\theta}_3) \end{array}$$

$C(X, Y) = E(X \cdot Y) - E(X) \cdot E(Y)$

$C(X, X) = E(X^2) - \left(E(X)\right)^2 = V(X)$

$\rho(\hat{\theta}_1, \hat{\theta}_2) < 0$ | because $\hat{\theta}_1 + \hat{\theta}_2 + \hat{\theta}_3 = 1$

$$\ell\ell_{\underline{c}}(\underset{\sim}{\theta} \mid \underset{\sim}{n}\, \mathcal{B}) = n_1 \log \theta_1 + \dots + n_k \log \theta_k$$

$$\boxed{\begin{aligned} \frac{\partial}{\partial \theta_1} \ell\ell_{\underline{c}}(\underset{\sim}{\theta} \mid \underline{n}\, \mathcal{B}) &= \frac{n_1}{\theta_1} \\ &\vdots \\ \frac{\partial}{\partial \theta_k} \ell\ell_{\underline{c}}(\underset{\sim}{\theta} \mid \underline{n}\, \mathcal{B}) &= \frac{n_k}{\theta_k} \end{aligned}}$$

linear constraint

$\cancel{g(\theta)} = \theta_1 + \dots + \theta_k = 1$

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unconstrainedly maximize

$$\mathcal{L}(\underset{\sim}{\theta}, \lambda) = \ell\ell_{\underline{c}}(\underset{\sim}{\theta} \mid \underset{\sim}{n}\, \mathcal{B}) - \lambda\, g(\underset{\sim}{\theta})$$

$$\boxed{-\lambda\left[\theta_1 + \dots + \theta_k - 1\right]}$$

$$\frac{\partial}{\partial \theta_1} \mathcal{L}(\underset{\sim}{\theta}, \lambda) = \frac{n_1}{\theta_1} - \lambda$$

$\vdots$

$$\frac{\partial}{\partial \theta_k} \mathcal{L}(\underset{\sim}{\theta}, \lambda) = \frac{n_k}{\theta_k} - \lambda$$

$$g(\theta_1, \dots, \theta_k) = \underline{\theta_1 + \dots + \theta_k - 1}$$

$$\begin{cases} \dfrac{n_1}{\theta_1} - \lambda = 0 \\ \vdots \\ \dfrac{n_k}{\theta_k} - \lambda = 0 \\ \theta_1 + .. + \theta_k = 1 \end{cases}$$

$$\frac{d}{d\lambda} \mathcal{L}(\underset{\sim}{\theta}, \lambda) = -(\theta_1 + \cdots + \theta_k - 1)$$

$(\lambda \neq 0)$

$$\begin{cases} n_1 = \lambda \theta_1 \\ \vdots \\ n_k = \lambda \theta_k \end{cases}$$

$$n = n_1 + \cdots + n_k = \lambda \underbrace{(\theta_1 + \cdots + \theta_k)}_{1}$$

$$\boxed{\hat{\theta}_1 = \frac{n_1}{n}, \ldots, \hat{\theta}_k = \frac{n_k}{n}}$$

$$\ell_c(\underset{\sim}{\theta} \mid \underset{\sim}{n}\ B) = \left(c \prod_{j=1}^{k} \theta_j^{n_j} \right)$$

$(\underset{\sim}{\alpha_j} > 0)$

$\underset{\sim}{\alpha} = (\alpha_1, \ldots, \alpha_k)$

$$p(\underset{\sim}{\theta} \mid \underset{\sim}{n}\ B) = c \prod_{j=1}^{k} \theta_j^{\alpha_j - 1}$$

Dirichlet$(\underset{\sim}{\alpha})$ PDF conjugate prior for Bernoulli / Binomial likelihood is $\mathrm{Beta}(\alpha, \beta)$ $\begin{pmatrix} \alpha > 0 \\ \beta > 0 \end{pmatrix}$

prior # of 1s

prior # of 0s

prior sample size $(\alpha + \beta)$

$0 < \theta < 1$

$$p(\theta \mid \alpha \, \beta \, [\mathrm{Beta}] \, \mathcal{B}) = c \, \theta^{\alpha - 1} (1 - \theta)^{\beta - 1} \quad (*)$$

$(k = 2)$

equivalent representation: $\underset{\sim}{\theta} = (\theta_1, \theta_2)$

$$p(\underset{\sim}{\theta} \mid \alpha \, \beta \, [\mathrm{Beta}] \, \mathcal{B}) = c \, \theta_1^{\alpha - 1} \, \theta_2^{\beta - 1} \quad (**)$$

$$= c \, \theta_1^{\alpha_1 - 1} \, \theta_2^{\alpha_2 - 1}$$

where $\theta_1 + \theta_2 = 1$

$\mathrm{Dirichlet}(\underset{\sim}{\alpha})$ is a multivariate generalization of $\mathrm{Beta}(\alpha, \beta)$ to $k \geq 2$

prior sample size is $\sum_{j=1}^{k} \alpha_j$

α_3 'other'

$\alpha_2 =$ 'Trump'

$\alpha_1 =$ # of prior votes for 'Biden'

LI Dirichlet($\underset{\sim}{\alpha}$) prior: all α_j positive but close to : $\underset{\sim}{\alpha} = (\epsilon, \epsilon, \ldots, \epsilon)$, for some $\epsilon > 0$ but (really) close to 0

$$p(\underset{\sim}{\theta} \mid \underset{\sim}{\alpha} \overset{\cancel{\underset{\sim}{N}}\ \underset{\sim}{n}}{\wedge} [\text{Dirichlet}]\ \mathcal{B}) =$$

$$c \cdot \underbrace{\left(c \prod_{j=1}^{k} \theta_j^{\alpha_j - 1} \right)}_{\text{prior}} \left(c \prod_{j=1}^{k} \theta_j^{n_j} \right)$$

$$= c \prod_{j=1}^{k} \theta_j^{(\alpha_j + n_j) - 1} = \text{Dirichlet}(\underset{\sim}{\alpha} + \cancel{\underset{\sim}{N}})$$

$$\left\{ \begin{array}{l} (\underset{\sim}{\theta} \mid \underset{\sim}{\alpha} [\text{Dirichlet}]\ \mathcal{B}) \sim \text{Dirichlet}(\underset{\sim}{\alpha}) \\ (\underset{\sim}{N} \mid \underset{\sim}{\theta}\ n\ \mathcal{B}) \sim \text{Multinomial}(n, \underset{\sim}{\theta}) \end{array} \right\} \to$$

$$\underset{\sim}{\theta} \mid \underset{\sim}{n} [\text{Dirichlet}]\ \mathcal{B}) \sim \text{Dirichlet}(\underset{\sim}{\alpha} + \underset{\sim}{n})$$