STAT 206
4 Feb 21
DD OH

1

$(\mathcal{Y}_i \mid \lambda\, \mathbb{E}\, \mathcal{B}) \overset{\text{IID}}{\sim} \text{Exponential}(\lambda)$ $(i = 1, \dots, n)$

for $\lambda > 0$ — in other words,

THT 2 problem 2(B)
(131)

(206)

$f_{\mathcal{Y}_i}(y_i \mid \lambda\, \mathbb{E}\, \mathcal{B}) = p_{\mathcal{Y}_i}(y_i \mid \lambda\, \mathbb{E}\, \mathcal{B})$

(206)

$$= p(y_i \mid \lambda\, \mathbb{E}\, \mathcal{B}) = \begin{cases} \frac{1}{\lambda} \exp\left(-\frac{y_i}{\lambda}\right) & \text{for } y_i > 0 \\ 0 & \text{else} \end{cases}$$

$$= \frac{1}{\lambda} \exp\left(-\frac{y_i}{\lambda}\right) \cdot I(y_i > 0)$$

(marginal PDF) $\to$ Mr. Fisher step $\boxed{1}$

$\boxed{2}$ (joint PDF) $p(\underset{\sim}{y} \mid \lambda\, \mathbb{E}\, \mathcal{B})$

$\underset{\sim}{y} = (y_1, \dots, y_n)$
$\underset{\sim}{\mathcal{Y}} = (\mathcal{Y}_1, \dots, \mathcal{Y}_n)$

$$= \prod_{i=1}^{n} p(y_i \mid \lambda\, \mathbb{E}\, \mathcal{B})$$

$$\circledast = \prod_{i=1}^{n} \frac{1}{\lambda} \exp\left(-\frac{y_i}{\lambda}\right) I(y_i > 0)$$

$$(*) = \frac{1}{\lambda^n} \exp\left(-\sum_{i=1}^{n} \frac{y_i}{\lambda}\right) I\left(\overset{\text{all}}{y_i} > 0\right)$$

$$= \lambda^{-n} \exp\left(-\frac{\sum_{i=1}^{n} y_i}{\lambda}\right) \quad \left(\begin{array}{c}\text{let's assume that} \\ \text{all } y_i > 0\end{array}\right)$$

$\boxed{3}$ define likelihood function :
& plot it

$$\ell(\lambda \mid y\, \mathbb{E}\, \mathcal{B}) \triangleq c \cdot p(y \mid \lambda\, \mathbb{E}\, \mathcal{B})$$

'ell'

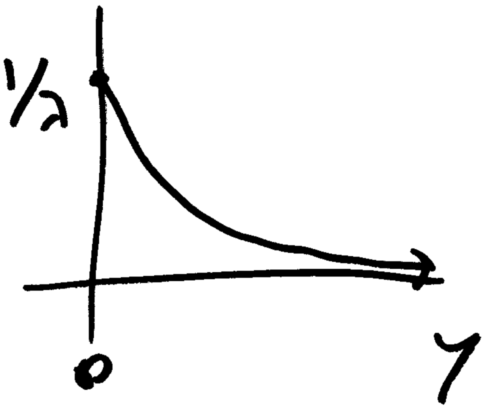
$\left(\begin{array}{c}\text{for any} \\ c > 0\end{array}\right)$
$$= c \cdot \lambda^{-n} \exp\left(-\frac{\sum_{i=1}^{n} y_i}{\lambda}\right)$$

$s \triangleq \sum_{i=1}^{n} y_i$
$$= c \cdot \lambda^{-n} \exp\left(-\frac{s}{\lambda}\right)$$

$\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$
$$= c \cdot \lambda^{-n} \exp\left(-\frac{n\bar{y}}{\lambda}\right)$$

---

in THT 1 problem 2(B)	$n = 14$	$s = 70612$ $\bar{y} = 5043.7$



$\frac{1}{\lambda} \exp\left(-\frac{y_i}{\lambda}\right)$

$\boxed{4}$ define log likelihood function & plot it

$$\ell\ell(\lambda \mid y\, \mathbb{E}\, \mathcal{B}) = -n \log \lambda - \frac{n \bar{y}}{\lambda}$$

$\boxed{5}$ find MLE

$$\frac{d}{d\lambda} \ell\ell(\lambda \mid y\, \mathbb{E}\, \mathcal{B}) = -\frac{n}{\lambda} + \frac{n\bar{y}}{\lambda^2}$$

$$= \frac{-n\lambda + n\bar{y}}{\lambda^2} = 0$$

iff $\lambda = \hat{\lambda}_{MLE} = \bar{y}$

$\boxed{6}$ find information & estimated $\hat{SE}(\hat{\lambda}_{MLE})$

$$\frac{d}{d\lambda} ll(\lambda \mid y \in \mathcal{B}) = -\frac{n}{\lambda} + \frac{n\bar{y}}{\lambda^2}$$

$$\frac{d^2}{d\lambda^2} ll(\lambda \mid y \in \mathcal{B}) = \frac{d}{d\lambda}\left(-n\lambda^{-1} + n\bar{y}\lambda^{-2}\right)$$

$$= \frac{n}{\lambda^2} - \frac{2n\bar{y}}{\lambda^3} = \frac{n\lambda - 2n\bar{y}}{\lambda^3}$$

$$\hat{I}(\hat{\lambda}_{MLE}) = \left[-\frac{d^2}{d\lambda^2} ll(\lambda \mid y \in \mathcal{B})\right]_{\lambda = \hat{\lambda}_{MLE}}$$

$$= -\frac{n\bar{y} - 2n\bar{y}}{\bar{y}^3} = \frac{n}{\bar{y}^2} = O(n) \checkmark$$

$$\hat{V}(\hat{\lambda}_{MLE}) = \left[\hat{I}(\hat{\lambda}_{MLE})\right]^{-1} = \frac{\bar{y}^2}{n}$$

$$\hat{SE}(\hat{\lambda}_{MLE}) = \sqrt{\hat{V}(\hat{\lambda}_{MLE})}$$

$$= \sqrt{\left[\hat{I}(\hat{\lambda}_{MLE})\right]^{-1}} = \frac{\bar{y}}{\sqrt{n}}$$

$$\doteq 1348.0$$

---

sanity checks

① $\hat{SE}(\text{anything}) \geq 0$ ✓ because $\bar{y} > 0$ here

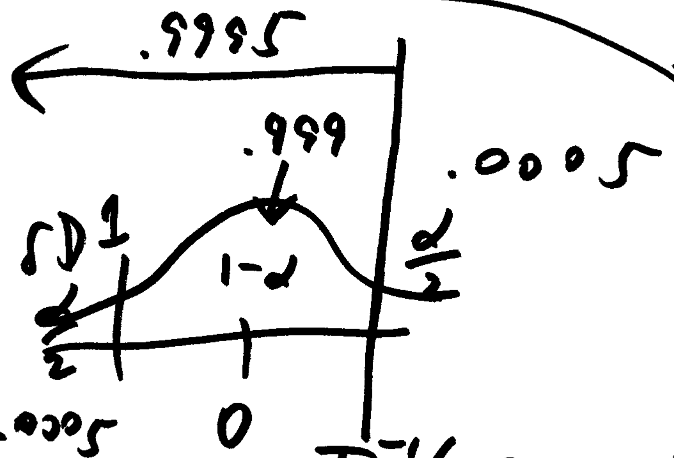
② in regular parametric problems,

$$\hat{SE}(\text{anything}) = \underline{O}\left(\frac{1}{\sqrt{n}}\right) \quad ✓$$

$\boxed{7}$ 99.9% large-n CI for λ : $(\alpha = .001)$

$$\boxed{\hat{\lambda}_{MLE} \pm \Phi^{-1}\left(1 - \frac{\alpha}{2}\right) \hat{SE}(\hat{\lambda}_{MLE})}$$

99.9% CI for $\hat{\lambda}$: (608.1, 9479.3) 6



$z_\alpha = \Phi^{-1}(.9995) = \text{qnorm}(.9995)$

as $n \downarrow$ left endpoint can go negative

or with $n = 14$

as $\alpha \downarrow 0$, left endpoint can also go negative

CLT issue: is $n = 14$ large enough

so that $\hat{\lambda}_{MLE} \underset{\text{repeated samples}}{\sim} N(\lambda, \cdot)$?