Length of stay (LOS) case study | STAT 206 17 Feb 21

(Lecture Notes part 6 pp. 77+) | DD AM discussion section

$y = (0, 1, \ldots, 6)$ $\quad n = 14$

full days in the hospital

A histogram (R) ① shows (using the cheating approach) that a Poisson sampling model is reasonable

① marginal PMF

for $\lambda > 0$ poisson here

$$p(y_i \mid \lambda \, \mathbb{P} \, \mathcal{B}) = \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} \cdot I(y_i = 0 \text{ or } 1 \text{ or } \ldots)$$

$\underbrace{}_{\cancel{\text{(assume = 1 here)}}}$

$y = (y_1, \ldots, y_n)$ $\quad (Y_i \mid \lambda \, \mathbb{P} \, \mathcal{B}) \stackrel{\text{IID}}{\sim} \text{Poisson}(\lambda)$

② joint PMF

$$p(y \mid \lambda \, \mathbb{P} \, \mathcal{B}) = \prod_{i=1}^{n} p(y_i \mid \lambda \, \mathbb{P} \, \mathcal{B})$$

$$= \prod_{i=1}^{n} \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} = \frac{\lambda^{y_1 + \cdots + y_n} e^{-n\lambda}}{\prod_{i=1}^{n} y_i!} = \frac{\lambda^{s} e^{-n\lambda}}{\prod_{i=1}^{n} y_i!}$$

with

$s = \sum_{i=1}^{n} y_i$

③ likelihood function:

(2)

$\ell(\lambda \mid y\, PB) = c \cdot \lambda^s e^{-n\lambda}$ for $\lambda > 0$

$c = \cancel{\prod_{i=1}^{n} y_i!}$

$(c = 1)$

(4) log likelihood $\ell\ell(\lambda \mid y\, PB) = s \log \lambda - n\lambda$

(5) MLE & suff. stat

$\frac{d}{d\lambda} \ell\ell(\lambda \mid y\, PB) = \frac{s}{\lambda} - n = 0$ iff

$\lambda = \hat{\lambda}_{MLE} = \frac{s}{n} = \bar{y} \doteq 2.1$ here

(*)

(6) visualize ℓ and $\ell\ell$ over an interesting range of λ values (R)

let's let the λ grid go from $(0.1, s_{17})$ to (s_{17}), $2\bar{y} \doteq 4.2$ (R)

sanity check on Poisson assumption

from (3)

$(Y \mid \lambda) \sim \text{Poisson}(\lambda) \rightarrow$

$E(Y) = V(Y) = \lambda$

so the sample mean $(\bar{y})$ and variance (s^2) of y should be close: $\left\{ \begin{array}{l} \bar{y} = 2.07 \\ s^2 = 2.38 \end{array} \right\}$

(looks reasonable)

(*) note that both s and $\bar{y} = \frac{s}{n}$ are (5) minimal sufficient for λ in this sampling model (also n, of course)

(6) observed information and standard error

$$\frac{d^2}{d\lambda^2} \ell\ell(\lambda \mid y\, \mathbb{P}\mathcal{B}) = -\frac{s}{\lambda^2}$$

so the observed information is

$$\hat{I}(\hat{\lambda}_{MLE}) = \left[-\frac{d^2}{d\lambda^2} \ell\ell(\lambda \mid y\, \mathbb{P}\mathcal{B}) \right]_{\lambda = \hat{\lambda}_{MLE}}$$

$$= \frac{s}{\hat{\lambda}_{MLE}^2} = \frac{s}{\left(\frac{s}{n}\right)^2} = \frac{n^2}{s} = \frac{n}{s/n} = \frac{n}{\hat{\lambda}_{MLE}} = \underline{O}(n) \checkmark$$

and the estimate standard error is then

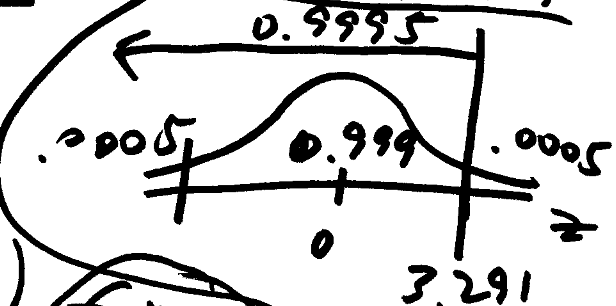
$$\widehat{SE}(\hat{\lambda}_{MLE}) = \sqrt{\hat{V}(\hat{\lambda}_{MLE})} = \sqrt{\hat{I}^{-1}(\hat{\lambda}_{MLE})}$$

$$= \sqrt{\frac{\hat{\lambda}_{MLE}}{n}} = \underline{O}\left(\frac{1}{\sqrt{n}}\right) \checkmark$$

(7) $100(1-\alpha)\% = 99.9\%$ CI for λ

CI: $\hat{\lambda}_{MLE} \pm \Phi^{-1}\left(1-\frac{\alpha}{2}\right) \cdot \hat{SE}\left(\hat{\lambda}_{MLE}\right)$ ④

CLT approx. (large n)



$2.071 \pm (3.291) \cdot (0.3847)$

full days in hospital

qnorm(.9995)

$= (0.81, 3.34)$

(could have gone negative with smaller α) "incoherent"

Bayesian analysis

①, ② ③ same as likelihood algorithm

$\ell(\lambda \mid \mathbb{P}\, \mathcal{B}) = c\, \lambda^{s} e^{-n\lambda}$

④ conjugate prior?

Q: familiar PDF?

Gelman et al. Appendix A

yes It's the family of $\boxed{\text{Gamma}}(\alpha, \beta)$ PDFs with $(\alpha > 0, \beta > 0)$

$p(\lambda \mid \Gamma\, \mathcal{B}) = c\, \lambda^{\alpha-1} e^{-\beta\lambda}\, I(\lambda > 0)$

⑤ ~~2ExpFam~~ product of 2 = another one? we have our conjugate prior yes

⑤ Graphically explore conjugate family ⑤

(R) α governs shape & center ($\alpha \uparrow$ skewness $\downarrow$) β governs spread in an inverse way:

$\Gamma(1, \beta) = \text{Exponential}(\beta)$

$\boxed{\lambda \sim \Gamma(\alpha, \beta) \quad E(\lambda) = \frac{\alpha}{\beta}}$ $\rightarrow$ as $\beta \uparrow$, $SD[\Gamma(\alpha, \beta)] \downarrow$

$V(\lambda) = \frac{\alpha}{\beta^2}$

$SD(\lambda) = \frac{\sqrt{\alpha}}{\beta}$

⑥ work out conjugate updating ~~(posterior)~~

$p(\lambda \mid y \, \Gamma \, P \, B) =$

(constant) $\cdot$ (prior) $\cdot$ (likelihood)

$C \cdot \boxed{p(\lambda \mid \Gamma \, B)} \cdot \ell(\lambda \mid y \, P \, B)$

$$= c \left(\lambda^{\alpha - 1} e^{-\beta \lambda} \right) \left(\lambda^{s} e^{-n \lambda} \right)$$

$$= c \, \lambda^{(\alpha + s) - 1} e^{-(\beta + n) \lambda} = \Gamma(\alpha + s, \beta + n)$$

prior PDF is $\Gamma(\alpha, \beta)$

likelihood PDF is $\lambda^{(s+1)-1} e^{-n\lambda} = \Gamma(s+1, n)$

6

Poisson(λ) family ($\lambda > 0$)

as $\lambda \uparrow$ { center $\uparrow$, spread $\uparrow$, shape becomes less skewed }

conjugate updating rule

$$\left\{ \begin{array}{l} (\lambda \mid \mathcal{C}\,\mathcal{B}) \sim \Gamma(\alpha, \beta) \\ (Y_i \mid \lambda\, \mathbb{P}\, \mathcal{B}) \overset{IID}{\sim} \text{Poisson}(\lambda) \\ \quad (i = 1, \dots, n) \end{array} \right\}$$

$s = \sum_{i=1}^{n} y_i$

$\mathbf{y} = (y_1, \dots, y_n)$

$$(\lambda \mid \mathbf{y}\, \mathcal{C}\, \mathbb{P}\, \mathcal{B}) \sim \underline{\Gamma(\alpha + s, \beta + n)}$$

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we're not *forced* to use a conj. prior (if it exists), but if a member of this family can be found that matches context $(\mathcal{C}, \mathcal{B})$,

$\boxed{\text{it's convenient to use it}}$ ex. LI ⑦

$\mathcal{I}(\alpha, \beta)$ prior

$$\left(\begin{array}{c}\text{posterior}\\ \text{mean}\end{array}\right) = \frac{(w_1)\left(\begin{array}{c}\text{prior}\\ \text{mean}\end{array}\right) + (w_2)\left(\begin{array}{c}\text{data}\\ \text{mean}\end{array}\right)}{(w_1) + (w_2)}$$

$$\frac{\alpha+s}{\beta+n} = \frac{(\beta)\left(\frac{\alpha}{\beta}\right) + (n)\left(\frac{s}{n}\right)}{(\beta) + (n)}$$

prior sample size $= n_{prior} = \beta$

data ——— $= n$

LI $\mathcal{I}(\alpha, \beta)$: take $\beta = \underline{\epsilon} > 0$ small positive $(\beta > 0)$

How specify α? for LI prior

prior mean $= \frac{\alpha}{\beta} = \frac{\alpha}{\epsilon}$ 0.001

if ϵ is close enough to 0, α could be just about any positive number

$(n = 14)$

$\frac{\alpha}{\epsilon} \doteq 5 \rightarrow \alpha = 5\epsilon = .005$

this will be a LI prior

sensitivity analysis: vary α/ϵ from $\{0.1, 1, 2, 5, 10\}$ 0.001

& superimpose all resulting posterior PDFs: should all be about same;

$\therefore$ precise specification of α with $\epsilon = 0.001$ doesn't matter much

utah $\frac{\alpha}{\epsilon} = 16$

class $\frac{\alpha}{\epsilon} = 5$